CHOOSING A NOMINAL SYSTEM PATH
TO MAXIMIZE THE PROBABILITY OF REACHING
A GIVEN REGION OF STATE SPACE*

by

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Summary

The statistical performance of a dynamical system is considered. Random initial conditions and/or random forcing functions cause the state variables of the system to be random variables rather than deterministic quantities. Nonetheless, a deterministic or nominal system path may be defined, and the performance of the system evaluated in terms of the nominal plus the statistics of the perturbations from the nominal. Under a "small" perturbation assumption, the perturbation equations are linearized. Adding the assumption of Gaussian input statistics, the state perturbations become Gaussian random variables. The probability that a given function of the terminal state lie within certain bounds may then be computed. Necessary conditions for the nominal initial conditions and/or control programs which maximize that probability are given. A straightforward gradient approach for numerical solution is sketched.

Description of the Problem

The system under consideration is assumed to satisfy a set of simultaneous first order ordinary differential equations

$$\dot{x} = f(x, u, w, t) \tag{1}$$

where

- x is an n-vector* of state variables
- u is an m-vector of control variables
- w is a p-vector of random forcing functions
- f is an n-vector of known functions of x, u, w, t
- t is the independent variable (usually time)
- $(\dot{\ }) = \frac{d}{dt}(\ )$

The system operates over a finite interval $t_o \le t \le t_f$. Because of the random forcing functions and/or random initial conditions, the state is a random vector. It is assumed that the control programs $u(t)$ will be the same for each operation of the system. The state history is written

$$x(t) = \bar{x}(t) + \delta x(t) \tag{2}$$

where $\bar{x}(t)$ is, by definition, the solution of (1) when there are no random perturbations. Thus, $\bar{x}(t)$ satisfies

$$\dot{\bar{x}} = f(\bar{x}, u, 0, t) \tag{3}$$

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* Vectors are column vectors with the exception that $\frac{\partial z}{\partial \beta}$, where z is a scalar and β is a vector, is a row vector. Hence, $\frac{\partial z}{\partial \beta}\delta\beta$ is a scalar product.

The initial conditions satisfy

$$x(t_o) = \bar{x}(t_o) + \delta x(t_o) \tag{4}$$

where $\bar{x}(t_o)$ may be partly given and partly free. The random component of $x(t_o)$ is $\delta x(t_o)$. In this analysis it is assumed that $\delta x(t_o)$ is a vector of Gaussian random variables, with given covariance

$$\mathcal{E}[\delta x(t_o)\, \delta x^T(t_o)] = X(t_o) \tag{5}$$

The random forcing functions $w(t)$ are assumed to be Gaussian white noise, with covariance

$$\mathcal{E}[w(t)\, w^T(\tau)] = Q(t)\, \delta(t - \tau) \tag{6}$$

The initial time t_o is assumed given. The terminal time t_f is determined from

$$\Omega[x(t_f), t_f] = 0 \tag{7}$$

where Ω is a known function of x and t. In particular, the nominal terminal time $\bar{t}_f$ is determined from (7) with $\bar{x}$ in place of x.

The problem is to choose the control programs $u(t)$ and those components of $\bar{x}(t_o)$ which are unspecified in order to maximize the probability that a given scalar function $\varphi[x(t_f), t_f]$ will lie between two values a and b. This definition of probability of success is only one possible problem statement. It does, however, appear more meaningful than many others and is mathematically tractable.

The Linearized Perturbation Approximation

The problem as described is meaningful and usually will have a solution. Unfortunately, there is no procedure available for evaluating the probability that $\varphi[x(t_f), t_f]$ lie between a and b besides the Monte Carlo approach, unless major simplifications are introduced. The crucial approximation of this (and many another) analysis is that the perturbations $\delta x(t)$ are always "small enough" that a negligible error is made by linearizing the perturbation equations:

$$\delta \dot{x} = \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial w} w \tag{8}$$

Because (8) is linear, and because $\delta x(t_o)$ and $w(t)$ are Gaussian, $\delta x(t)$ will also be a vector of Gaussian random variables. Under this condition the problem as stated can be solved.

The smallness assumption restricts the magnitudes of $X(t_o)$, $Q(t)$, $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial w\, \partial x}, \frac{\partial^2 f}{\partial x\, \partial u}, \frac{\partial^2 f}{\partial w\, \partial u}$*. (Because of assumption (6) it is necessary to assume that (1) is linear in w ($\frac{\partial^2 f}{\partial w^2} = 0$). The square of a white noise component is not integrable.) There is no simpler way to state the restriction than to say that (8) must produce $\delta x(t)$ with negligible error.

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* The second partials are merely symbolic expressions for third order tensors such as $\frac{\partial^2 f_i}{\partial x_j\, \partial x_k}$, $i, j, k = 1, ---, n$.

Evaluation of the Probability of Success

The function $\varphi[x(t_f), t_f]$ may be written

$$\varphi[x(t_f), t_f] = \varphi[\bar{x}(\bar{t}_f), \bar{t}_f] + d\varphi$$

$$= \bar{\varphi} + d\varphi \tag{9}$$

To first order, consistent with (8),

$$d\varphi = \left(\frac{\partial\varphi}{\partial x}\right)_{t=\bar{t}_f} \delta x(\bar{t}_f) + \dot{\varphi}\, dt_f \tag{10}$$

where

$$\dot{\varphi} = \left[\frac{\partial\varphi}{\partial x}\dot{x} + \frac{\partial\varphi}{\partial t}\right]_{t=\bar{t}_f}$$

and dt_f is determined from

$$0 = d\Omega = \left(\frac{\partial\Omega}{\partial x}\right)_{t=\bar{t}_f} \delta x(\bar{t}_f) + \dot{\Omega}\, dt_f \tag{11}$$

Substituting for dt_f from (11)

$$d\varphi = \left[\frac{\partial\varphi}{\partial x} - \frac{\dot{\varphi}}{\dot{\Omega}}\frac{\partial\Omega}{\partial x}\right]_{t=\bar{t}_f} \delta x(\bar{t}_f) \tag{12}$$

Because $\delta x(\bar{t}_f)$ is a vector of Gaussian random variables, $d\varphi$ is a Gaussian scalar. The probability that $\varphi[x(t_f), t_f]$ lie between a and b is

$$J = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{a-\bar{\varphi}}^{b-\bar{\varphi}} e^{-y^2/2\sigma^2}\, dy \tag{13}$$

where

$$\sigma^2 = \mathcal{E}[(d\varphi)^2] \tag{14}$$

σ^2 may be evaluated by integrating $\frac{d}{dt}\mathcal{E}[\zeta^T \delta x \; \delta x^T \zeta]$, with

$$\dot{\zeta}^T + \zeta^T \frac{\partial f}{\partial x} = 0 \tag{15}$$

$$\zeta^T(\bar{t}_f) = \left[\frac{\partial \varphi}{\partial x} - \frac{\dot{\varphi}}{\dot{\Omega}} \frac{\partial \Omega}{\partial x}\right]_{t=\bar{t}_f} \tag{16}$$

This gives

$$\sigma^2 = [\zeta^T X \zeta]_{t=t_o} + \int_{t_o}^{\bar{t}_f} \zeta^T \frac{\partial f}{\partial w} Q \left(\frac{\partial f}{\partial w}\right)^T \zeta \, d\tau \tag{17}$$

Equation (17) has used the noise statistics (6) and the relation

$$\mathcal{E}[\delta x(t) \, w^T(t)] = \frac{1}{2} \frac{\partial f}{\partial w}(t) \, Q(t) \tag{18}$$

Necessary Conditions for an Extremal Solution

A standard variational approach is to adjoin the constraint equations to the performance index by use of Lagrange multipliers. In this problem the constraint equations are (3) and (15). The augmented J is written

$$J = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{a-\bar{\varphi}}^{b-\bar{\varphi}} e^{-y^2/2\sigma^2} \, dy$$

$$+ \int_{t_o}^{\bar{t}_f} \{\lambda^T(\bar{f} - \dot{\bar{x}}) + \ell^T[(\frac{\partial f}{\partial x})^T \zeta + \dot{\zeta}]\} d\tau \tag{19}$$

where $\lambda(t)$ and $\ell(t)$ are each n-vectors of Lagrange multiplier functions.

If J is to be a relative maximum, it must be stationary with respect to arbitrary small perturbations in $\bar{x}(t_o)$ and/or $u(t)$. A small change in J is given by

$$dJ = -\frac{1}{\sqrt{2\pi}\,\sigma^2}\left[\int_{a-\bar{\varphi}}^{b-\bar{\varphi}} e^{-y^2/2\sigma^2}\left(1+\frac{y^2}{\sigma^2}\right)dy\right]d\sigma$$

$$-\frac{1}{\sqrt{2\pi}\,\sigma}\left[e^{-(b-\bar{\varphi})^2/2\sigma^2} - e^{-(a-\bar{\varphi})^2/2\sigma^2}\right]d\bar{\varphi}$$

$$+\int_{t_o}^{\bar{t}_f} \lambda^T\left(\frac{\partial f}{\partial x}\delta\bar{x} + \frac{\partial f}{\partial u}\delta u - \delta\dot{\bar{x}}\right)d\tau$$

$$+\int_{t_o}^{\bar{t}_f} \left\{\frac{\partial}{\partial x}\left[\ell^T\left(\frac{\partial f}{\partial x}\right)^T\zeta\right]\delta\bar{x} + \frac{\partial}{\partial u}\left[\ell^T\left(\frac{\partial f}{\partial x}\right)^T\zeta\right]\delta u + \ell^T\left(\frac{\partial f}{\partial x}\right)^T\delta\zeta + \ell^T\delta\dot{\zeta}\right\}d\tau \tag{20}$$

Equation (20) is the change in J due to changes in the nominal path. $d\sigma$ may be determined from (17):

$$2\sigma\, d\sigma = 2(\zeta^T X\,\delta\zeta)_o + \int_{t_o}^{\bar{t}_f}\left\{2\zeta^T\frac{\partial f}{\partial w}Q\left(\frac{\partial f}{\partial w}\right)^T\delta\zeta + \frac{\partial}{\partial x}\left[\zeta^T\frac{\partial f}{\partial w}Q\left(\frac{\partial f}{\partial w}\right)^T\zeta\right]\delta\bar{x}\right.$$

$$\left. + \frac{\partial}{\partial u}\left[\zeta^T\frac{\partial f}{\partial w}Q\left(\frac{\partial f}{\partial w}\right)^T\zeta\right]\delta u\right\}d\tau \tag{21}$$

$d\bar{\varphi}$ is obtained directly from (12) by using $\delta\bar{x}(\bar{t}_f)$ in place of $\delta x(\bar{t}_f)$.

The $\delta\dot{\bar{x}}$ and $\delta\dot{\zeta}$ terms in (20) are integrated by parts, in the usual manner. Finally, dJ may be written as

$$dJ = \left(\frac{\alpha}{\sigma}\zeta^T X - \ell^T\right)_{t=t_o}\delta\zeta(t_o) + \left(\lambda^T\delta\bar{x}\right)_{t=t_o} + \Big[\beta\left(\frac{\partial\varphi}{\partial x} - \frac{\dot{\varphi}}{\dot{\Omega}}\frac{\partial\Omega}{\partial x}\right) - \lambda^T$$

$$+ \ell^T\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x} - \frac{\dot{\varphi}}{\dot{\Omega}}\frac{\partial\Omega}{\partial x}\right)^T\Big]_{t=t_f}\delta\bar{x}(\bar{t}_f) + \int_{t_o}^{\bar{t}_f}\{\dot{\lambda}^T + \lambda^T\frac{\partial f}{\partial x}$$

$$+ \frac{\partial}{\partial x}[\ell^T(\frac{\partial f}{\partial x})^T\zeta] + \frac{\alpha}{2\sigma}\frac{\partial}{\partial x}[\zeta^T\frac{\partial f}{\partial w}Q(\frac{\partial f}{\partial w})^T\zeta]\}\delta\bar{x}(\tau)d\tau$$

$$+ \int_{t_o}^{\bar{t}_f}\{\lambda^T\frac{\partial f}{\partial u} + \frac{\partial}{\partial u}[\ell^T(\frac{\partial f}{\partial x})^T\zeta] + \frac{\alpha}{2\sigma}\frac{\partial}{\partial u}[\zeta^T\frac{\partial f}{\partial w}Q(\frac{\partial f}{\partial w})^T\zeta]\}\delta u(\tau)d\tau$$

$$+ \int_{t_o}^{\bar{t}_f}\{-\dot{\ell}^T + \ell^T(\frac{\partial f}{\partial x})^T + \frac{\alpha}{\sigma}\zeta^T\frac{\partial f}{\partial w}Q(\frac{\partial f}{\partial w})^T\}\delta\zeta(\tau)d\tau \qquad (22)$$

where

$$\alpha = -\frac{1}{\sqrt{2\pi}\,\sigma^2}\int_{a-\bar{\varphi}}^{b-\bar{\varphi}} e^{-y^2/2\sigma^2}\left(1 + \frac{y^2}{\sigma^2}\right)dy$$

$$\beta = -\frac{1}{\sqrt{2\pi}\,\sigma}\left[e^{-(b-\bar{\varphi})^2/2\sigma^2} - e^{-(a-\bar{\varphi})^2/2\sigma^2}\right]$$

Necessary conditions for stationary J include

$$\dot{\lambda}^T + \lambda^T\frac{\partial f}{\partial x} + \frac{\partial}{\partial x}[\zeta^T\frac{\partial f}{\partial x}\ell] + \frac{\alpha}{2\sigma}\frac{\partial}{\partial x}[\zeta^T\frac{\partial f}{\partial w}Q(\frac{\partial f}{\partial w})^T\zeta] = 0 \qquad (23)$$

$$\lambda^T(\bar{t}_f) = \left[\beta\left(\frac{\partial\varphi}{\partial x} - \frac{\dot{\varphi}}{\dot{\Omega}}\frac{\partial\Omega}{\partial x}\right) + \ell^T \frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x} - \frac{\dot{\varphi}}{\dot{\Omega}}\frac{\partial\Omega}{\partial x}\right)^T\right]_{t=\bar{t}_f} \tag{24}$$

$$\dot{\ell}^T - \ell^T\left(\frac{\partial f}{\partial x}\right)^T - \frac{\alpha}{\sigma}\zeta^T \frac{\partial f}{\partial w} Q\left(\frac{\partial f}{\partial w}\right)^T = 0 \tag{25}$$

$$\ell^T(t_o) = \frac{\alpha}{\sigma}(\zeta^T X)_{t=t_o} \tag{26}$$

$$\lambda^T \frac{\partial f}{\partial u} + \frac{\partial}{\partial u}[\zeta^T \frac{\partial f}{\partial x}\ell] + \frac{\alpha}{2\sigma}\frac{\partial}{\partial u}[\zeta^T \frac{\partial f}{\partial w} Q\left(\frac{\partial f}{\partial w}\right)^T \zeta] = 0 \tag{27}$$

Each term in the scalar product $(\lambda^T \delta\bar{x})_o$ is zero either because the λ_o or the $\delta\bar{x}_o$ component is zero.

The necessary conditions are the coupled differential equations for $\bar{x}, \zeta, \lambda, \ell$ with mixed boundary conditions, and (27) which gives the extremalizing $u(t)$ as a function of x, ζ, λ and ℓ.

One rather awkward possibility has been omitted in obtaining dJ. This is the dependence of $\frac{\dot{\varphi}}{\dot{\Omega}}$ upon $u(\bar{t}_f)$. If there is such dependence, some constraint must be placed on $\dot{u}$ in the neighborhood of t_f in order to avoid violating the continuity assumptions implicit in the dJ derivation. Refs. [1] and [2] make attempts at imposing sensible constraints. This problem is really too specialized to discuss further in this paper.

Simple Example Problem

Consider the one variable system which satisfies

$$\dot{x} = -x + x^2 \tag{28}$$

Suppose that the problem is to maximize the probability that $|x(1)| \leq a$. $\bar{x}(0)$ is free to be selected, $\delta x(0)$ is a Gaussian random variable with variance σ_o^2. There is no control variable u, and $\Omega = t - 1$ $(t_o = 0)$. The linearized perturbation approximation in this case requires that $|x|$ be always much less than one. Hence, for the problem to be interesting, a must be much less than one.

Equation (28) can be integrated analytically. The solution (for $x_o < 1$) is

$$\frac{x}{1-x} = \frac{x_o}{1-x_o} e^{-t} \tag{29}$$

where $x(0) = x_o$. With solution (29), one can relate $x = \pm a$ to the value of x_o which produces it. Hence, the probability that $-a \leq x(1) \leq a$ is equal to the probability that $-b_2 \leq x_o \leq b_1$. For $a = .05$, $b_1 = .120$, $b_2 = .158$, the probability that x_o lie between $-.158$ and $+.120$ is

$$\frac{1}{\sqrt{2\pi}\,\sigma} \int_{-.158-\bar{x}_o}^{.120-\bar{x}_o} e^{-y^2/2\sigma_o^2}\, dy \tag{30}$$

since the integration is over the probability distribution of δx_o, and $\delta x_o = x_o - \bar{x}_o$. For this example there is no need to use the necessary conditions of the previous section because of the analytic solution (29). In fact, it is obvious from the nature of $e^{-y^2/2\sigma_o^2}$ that the value of $\bar{x}_o$ which maximizes (30) is the one which makes the lower limit the negative of the upper limit. Hence, $\bar{x}_o = -.019$ is the extremalizing choice, for $a = .05$ and for any σ_o.

It is readily observable that $x \to \infty$ in finite time if $x_o > 1$. Hence, b_1 must be less than one for the above computational procedure to be valid. Without solution (29), however, it would also be necessary to limit σ_o so that a sufficiently accurate approximation could be carried out.

For the system governed by (28), the perturbation equation before linearization is

$$\delta\dot{x} = -(1 - 2\bar{x})\delta x + (\delta x)^2 \tag{31}$$

In order to use (8), the "smallness" assumption is

$$(\delta x)^2 << (1 - 2\bar{x}) \tag{32}$$

Thus, it is easy to observe the requirement that

$$\sigma_o^2 << (1 - 2\bar{x}_o) \tag{33}$$

It is apparent that $|\bar{x}_o| << 1$, so (33) requires $\sigma_o^2 << 1$.

Computing the Gradient of J for Iterative Optimization

The essence of a gradient method is calculation of the relation between changes in the performance index and changes in the control variables or parameters. Here, $u(t)$ and/or $\bar{x}(t_o)$ are the controls. The steps in a straightforward gradient procedure are as follows:

1. Choose $\bar{x}(t_o)$ and $u(t)$, calculate $\bar{x}(t)$ from (3), with $\bar{t}_f$ determined by (7). Calculate and store $\frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial f}{\partial w}, \frac{\partial^2 f}{\partial x \partial w}, \frac{\partial^2 f}{\partial u \partial x}, \frac{\partial^2 f}{\partial u \partial w}$.

2. Calculate and store $\zeta(t)$ using (16) and (15). Calculate σ from (17). Calculate α and β.

3. Calculate and store $\ell(t)$ using (26) and (25).

4. Calculate $\lambda(t)$ using (24) and (23), and simultaneously calculate and store the left hand side of (27), which will not be zero.

By following these steps, the predicted change in J due to a "small" change in $\bar{x}(t_o)$ and/or $u(t)$ is

$$dJ = (\lambda^T \delta\bar{x})_{t=t_o} + \int_{t_o}^{\bar{t}_f} \{\lambda^T \frac{\partial f}{\partial u} + \frac{\partial}{\partial u}[\zeta^T \frac{\partial f}{\partial x} \ell] + \frac{\alpha}{2\sigma} \frac{\partial}{\partial u}[\zeta^T \frac{\partial f}{\partial w} Q (\frac{\partial f}{\partial w})^T \zeta]\} \delta u(\tau)\, dt \qquad (34)$$

The gradient of J with respect to $\bar{x}(t_o)$ and $u(t)$ is clear from (34). References [3, 4, 5] describe procedures for using the gradient to improve the choices of $\bar{x}(t_o)$ and $u(t)$.

It should be observed that any numerical scheme which has been applied to deterministic optimization can (at least conceptually) be used for maximizing J of this paper. Given the input statistics, J becomes a deterministic quantity and may be treated with deterministic optimization techniques.

Joint Probabilities

Suppose that the desired objective is to maximize the joint probability that $\varphi_1[x(t_f), t_f]$ be between a_1 and b_1 and that $\varphi_2[x(t_f), t_f]$ be between a_2 and b_2. Arguing as before, $d\varphi_1$ and $d\varphi_2$ are both Gaussian random variables, and the joint probability of interest may be written as

$$J = \frac{1}{2\pi |P|^{1/2}} \int_{a_1 - \bar{\varphi}_1}^{b_1 - \bar{\varphi}_1} dy \int_{a_2 - \bar{\varphi}_2}^{b_2 - \bar{\varphi}_2} dz \; \exp\{-\tfrac{1}{2}[y^T z^T] P^{-1} [\begin{smallmatrix} y \\ z \end{smallmatrix}]\} \tag{35}$$

where

$$P_{ij} = [\zeta_i^T X \zeta_j]_{t=t_o} + \int_{t_o}^{t_f} \zeta_i^T \frac{\partial f}{\partial w} Q \left(\frac{\partial f}{\partial w}\right)^T \zeta_j \, dt \qquad i, j = 1, 2 \tag{36}$$

where ζ_i is the solution of (15) using boundary condition (16) with φ_i in place of φ. Both theoretically and computationally, two $\zeta(t)$ solutions are required. Further, two $\ell(t)$ solutions are required. The entire development may be carried through by analogy to produce dJ as a function of $\delta\bar{x}(t_o)$ and $\delta u(t)$. The necessary conditions for stationary J are simply extended forms of the ones previously given.

Finally, it is clear that the number of functions of the terminal conditions that may be considered may be as many as, but not more than, n, the number of state variables. J becomes an integral over the joint distribution of the Gaussian random variables $d\varphi_1$, $d\varphi_2$, - - -, $d\varphi_k$, $k \le n$. There will be $2k$ expressions similar to, but more complex than, the α and β expressions. The basic approach, however, both conceptually and numerically, is unchanged with k larger than one.

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